Indian Statistical Institute

Mid-Semestral Examination 2013-2014

M.Math First Year

Linear Algebra

Time : 3 Hours Date : 10.09.2013 Maximum Marks : 100 Instructor : Jaydeb Sarkar

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(I) Answer all questions.

(II) Unless specified otherwise, V is a vector space over a field F and dim$V < \infty$.

Q1. (15 marks) Let $m \geq 2$ be an integer and $\{V_i\}_{i=1}^m$ be a sequence of finite dimensional vector spaces over a field F. Also let $V_0 = V_{m+1} = \{0\}$ and $T_i \in \mathcal{L}(V_i, V_{i+1})$ for each $i = 0, 1, \ldots, m$, where T_0 and T_m are the zero maps. Furthermore, assume that $\text{ran}T_i = \ker T_{i+1}$ for all $i = 0, \ldots, m-1$. Prove that

$$\sum_{i=1}^{m} (-1)^i \dim V_i = 0.$$

Q2. (2+5+8 marks) Let $V \subseteq \mathbb{R}[x]$ be the set of all polynomials of degree ≤ 2.

(i) Prove that $\mathcal{B} = \{1, x, x^2\}$ is a basis for V.

(ii) Determine the dual basis $\mathcal{B}^*$ of $\mathcal{B}$.

(iii) For each $p \in V$, define $Tp \in V^*$ by

$$(Tp)(q) = \int_{-1}^{1} p(x)q(x)\, dx. \qquad (\forall q \in V)$$

Prove that $T \in \mathcal{L}(V, V^*)$ and find $[T]_{\mathcal{B}}^{\mathcal{B}^*}$.

Q3. (3+4+8 marks) Consider the map $ev_0 : \mathbb{R}[x] \to \mathbb{R}$, defined by $ev_0(p) = p(0)$ for all $p \in \mathbb{R}[x]$. Prove that

(i) ev_0 is a linear map.

(ii) ev_0 is right invertible.

(iii) ev_0 is not left invertible (do not use part (ii)).

Q4. (15 marks) Does there exists a basis for $\mathcal{L}(V)$ consisting invertible operators? Justify your answer.

Q5. (15 marks) Let dim $V = n > 1$, and $T_1, T_2 \in \mathcal{L}(V)$. Let $T_1^n = T_2^n = 0$ but $T_1^{n-1} \neq 0$ and $T_2^{n-1} \neq 0$. Prove that T_1 and T_2 are similar.

Q6. (6+9 marks) (i) Let $T \in \mathcal{L}(V)$ and dim $V = n$. Prove that, if $T^m = 0$ for some positive integer m, then $T^n = 0$.

(ii) Let $T_1, T_2 \in \mathcal{L}(V)$. Prove that $T_2 = XT_1$ for some $X \in \mathcal{L}(V)$ if and only if $\ker T_1 \subseteq \ker T_2$.

Q7. (10 + 5 marks) (a) Let $A \in M_3(\mathbb{R})$ and let A is not similar, over $\mathbb{R}$, to a triangular matrix. Prove that A is diagonalizable over $\mathbb{C}$.

(b) Let $A \in M_n(F)$ and that $A^2 = A$. Prove that A is similar to a diagonal matrix.